Priority Queues (Heaps)

Motivation

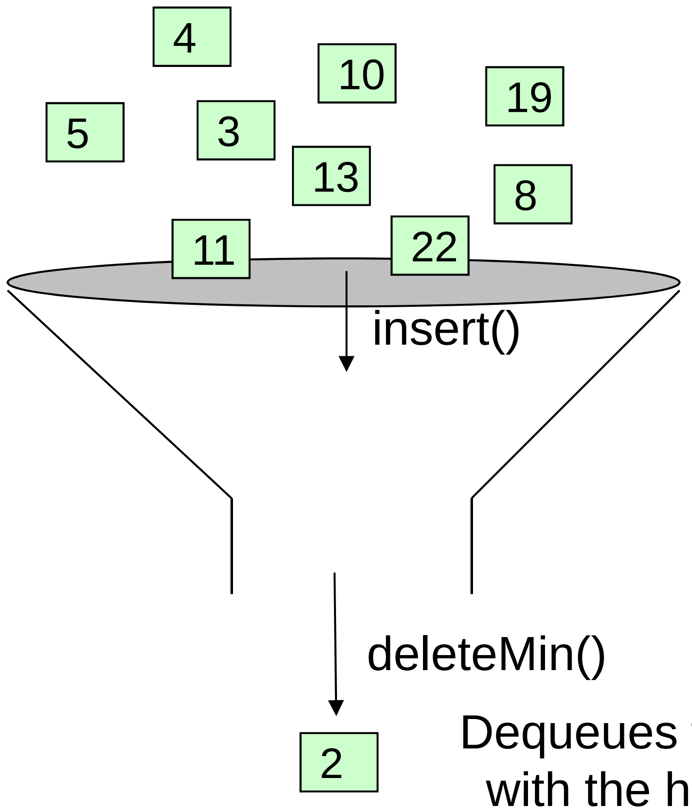
- Queues are a standard mechanism for ordering tasks on a first-come, first-served basis
- However, some tasks may be more important or timely than others (higher priority)
- <u>Priority queues</u>
  - Store tasks using a partial ordering based on priority
  - Ensure highest priority task at head of queue
- <u>Heaps</u> are the underlying data structure of priority queues

Priority Queues: Specification

- Main operations
  - **`insert`** (i.e., enqueue)
    - Dynamic insert
    - specification of a priority level (0-high, 1,2.. Low)
  - **`deleteMin`** (i.e., dequeue)
    - Finds the current minimum element (read: "highest priority") in the queue, deletes it from the queue, and returns it

- Performance goal is for operations to be "fast"

Using priority queues

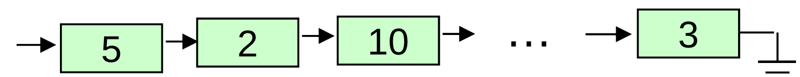


Dequeues the next element with the highest priority

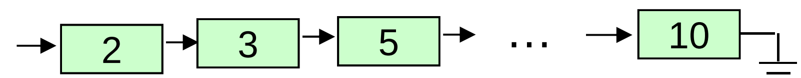
Can we build a data structure better suited to store and retrieve priorities?

Simple Implementations

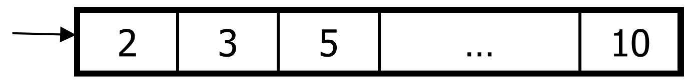
- Unordered linked list
  - O(1) insert
  - O(n) deleteMin



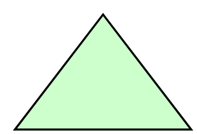
- Ordered linked list
  - O(n) insert
  - O(1) deleteMin



- Ordered array
  - O(lg n + n) insert
  - O(n) deleteMin



- Balanced BST
  - O($\log_2$n) insert and deleteMin



Binary Heap

A priority queue data structure

Binary Heap

- A binary heap is a binary tree with two properties
  - Structure property
  - Heap-order property

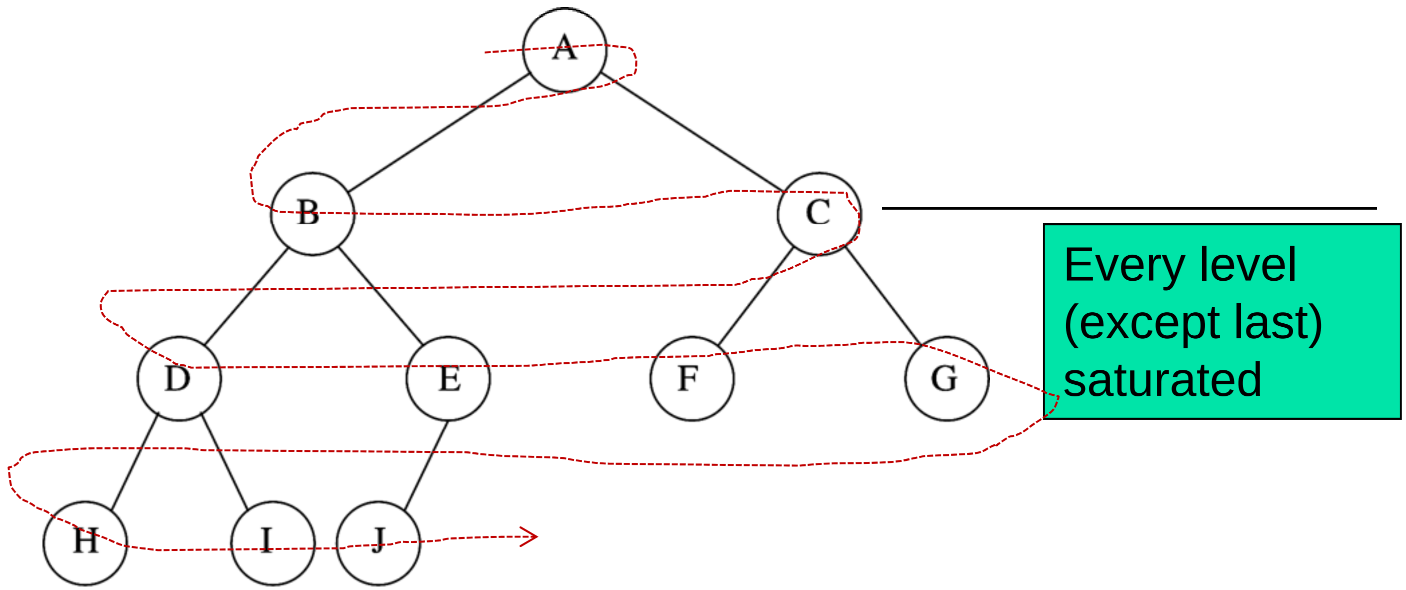
Structure Property

- A binary heap is a complete binary tree
  - Each level (except possibly the bottom most level) is completely filled
  - The bottom most level may be partially filled (from left to right)

- Height of a complete binary tree with N elements is $\lfloor \log_2 N \rfloor$

Binary Heap Example

N=10



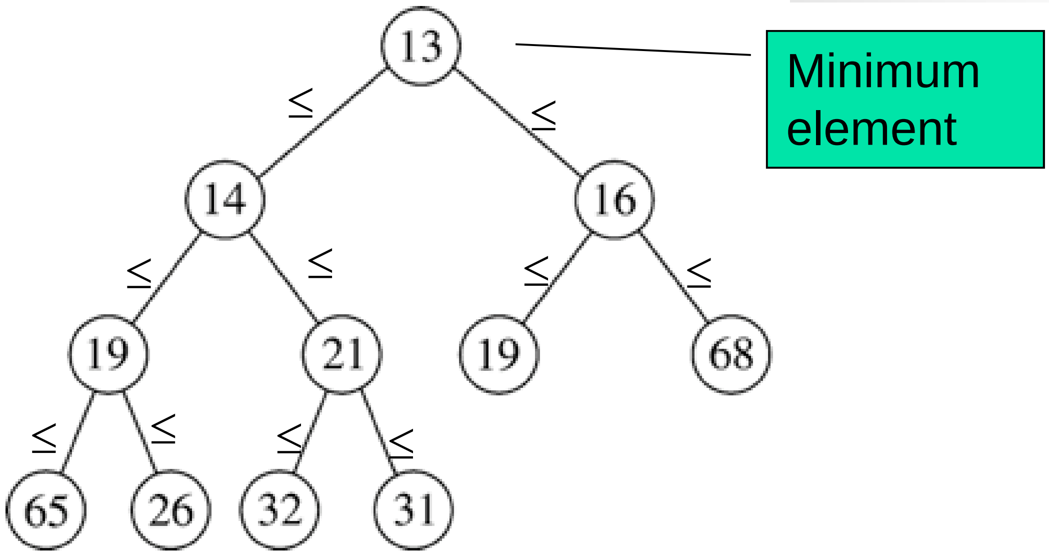
Array representation:

0	1	2	3	4	5	6	7	8	9	10	11	12	13
	A	B	C	D	E	F	G	H	I	J			

Heap-order Property

- Heap-order property (for a "MinHeap")
  - For every node X, key(parent(X)) $\leq$ key(X)
  - Except root node, which has no parent
- Thus, minimum key always at root
  - Alternatively, for a "MaxHeap", always keep the maximum key at the root
- Insert and deleteMin must maintain heap-order property

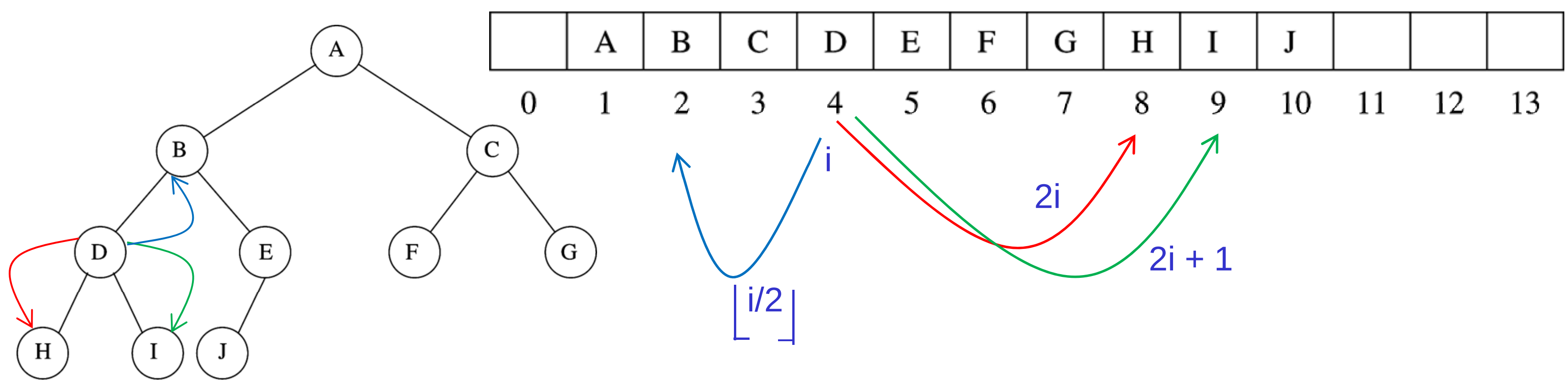
Heap Order Property



- Duplicates are allowed
- No order implied for elements which do not share ancestor-descendant relationship

Implementing Complete Binary Trees as Arrays

- Given element at position i in the array
  - Left child(i) = at position 2i
  - Right child(i) = at position 2i + 1
  - Parent(i) = at position $\lfloor i/2 \rfloor$



```cpp
template <typename Comparable>
class BinaryHeap
{
  public:
    explicit BinaryHeap( int capacity = 100 );
    explicit BinaryHeap( const vector<Comparable> & items );

    bool isEmpty( ) const;
    const Comparable & findMin( ) const;

    void insert( const Comparable & x );
    void deleteMin( );
    void deleteMin( Comparable & minItem );
    void makeEmpty( );

  private:
    int                currentSize;  // Number of elements in heap
    vector<Comparable> array;        // The heap array

    void buildHeap( );
    void percolateDown( int hole );
};
```

Just finds the Min without deleting it

insert

deleteMin

Note: a general delete() function is not as important for heaps but could be implemented

Stores the heap as a vector

Fix heap after deleteMin

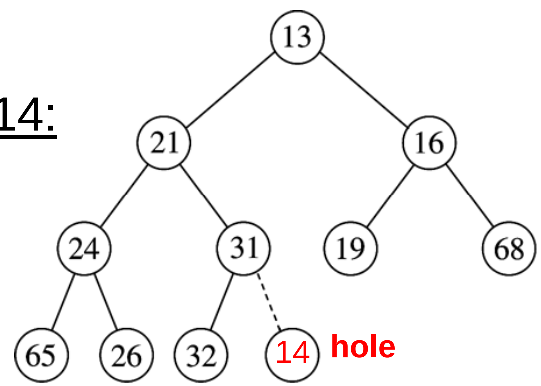
Heap Insert

- Insert new element into the heap at the next available slot ("hole")
  - According to maintaining a complete binary tree
- Then, "percolate" the element up the heap while heap-order property not satisfied

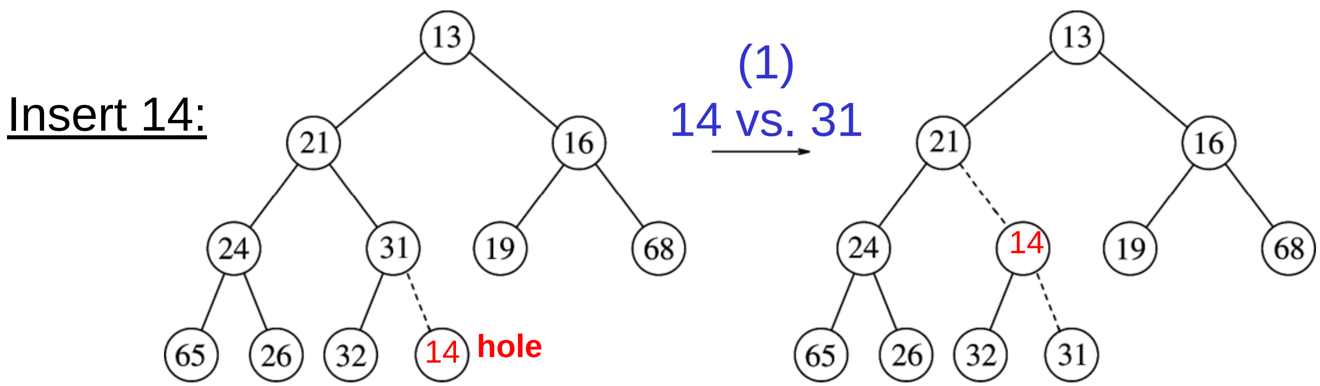
Percolating Up

Heap Insert: Example

Insert 14:

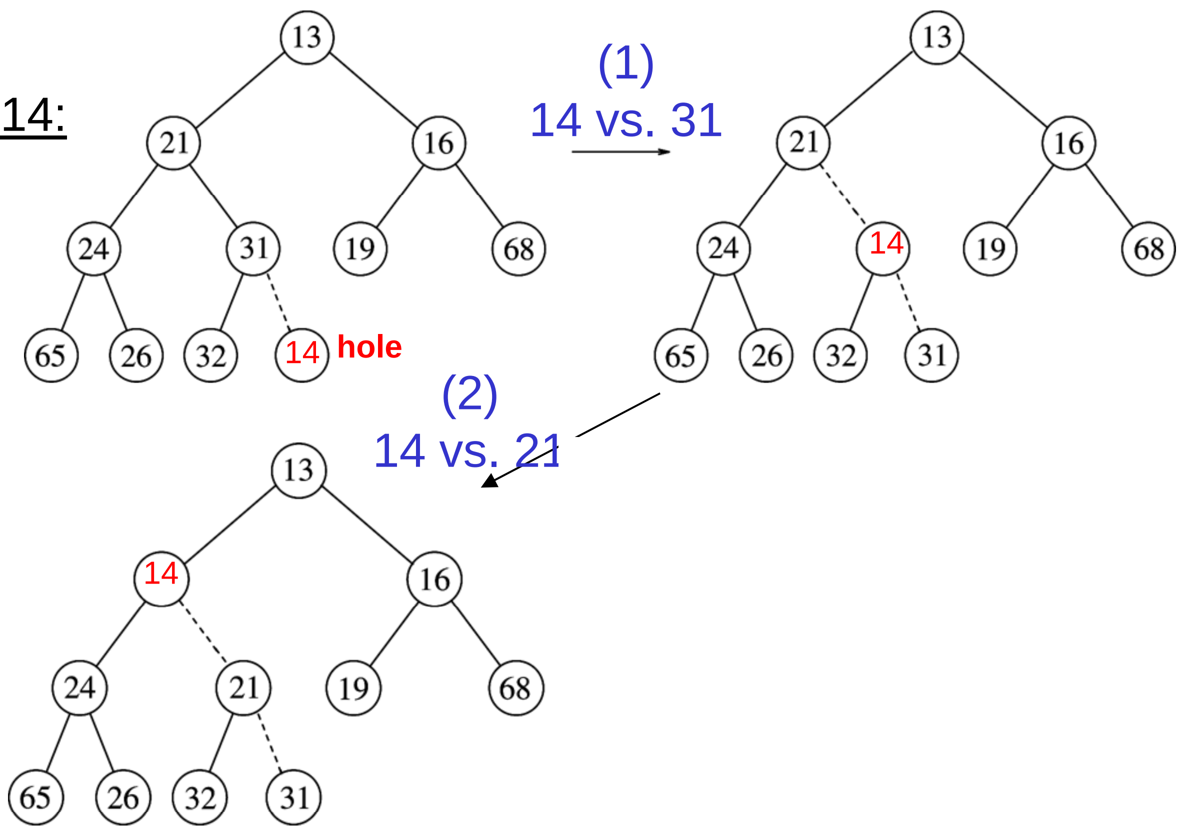


Heap Insert: Example



Heap Insert: Example

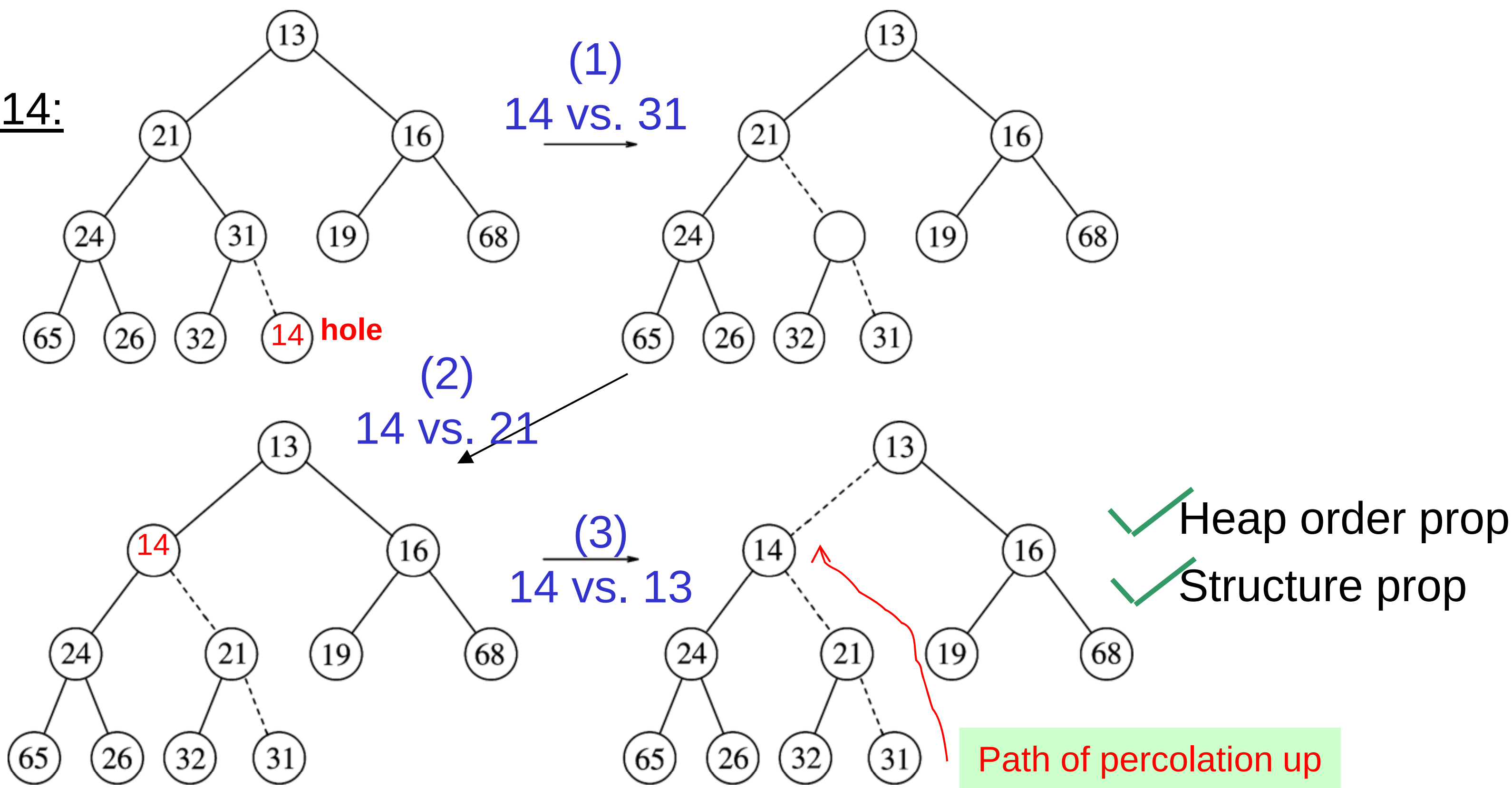
Insert 14:



Percolating Up

Heap Insert: Example

<u>Insert 14:</u>



Heap Insert: Implementation

```
// assume array implementation
void insert( const Comparable &x) {
?
}
```

Heap Insert: Implementation

```
/**
 * Insert item x, allowing duplicates.
 */
void insert( const Comparable & x )
{
    if( currentSize == array.size( ) - 1 )
        array.resize( array.size( ) * 2 );

        // Percolate up
    int hole = ++currentSize;
    for( ; hole > 1 && x < array[ hole / 2 ]; hole /= 2 )
        array[ hole ] = array[ hole / 2 ];
    array[ hole ] = x;
}
```

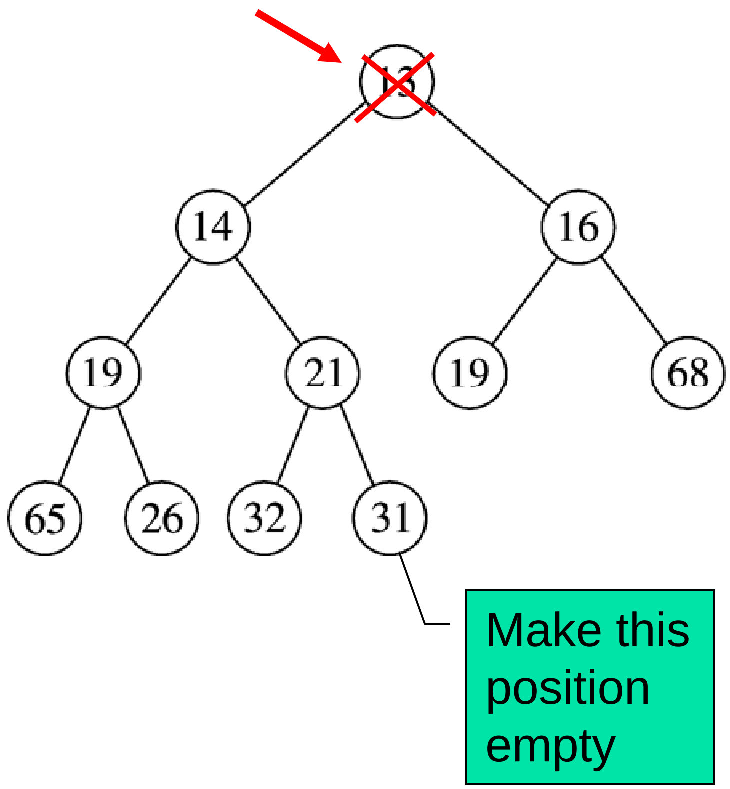
O(log N) time

Heap DeleteMin

- Minimum element is always at the root
- Heap decreases by one in size
- Move last element into hole at root
- ***Percolate down*** while heap-order property not satisfied

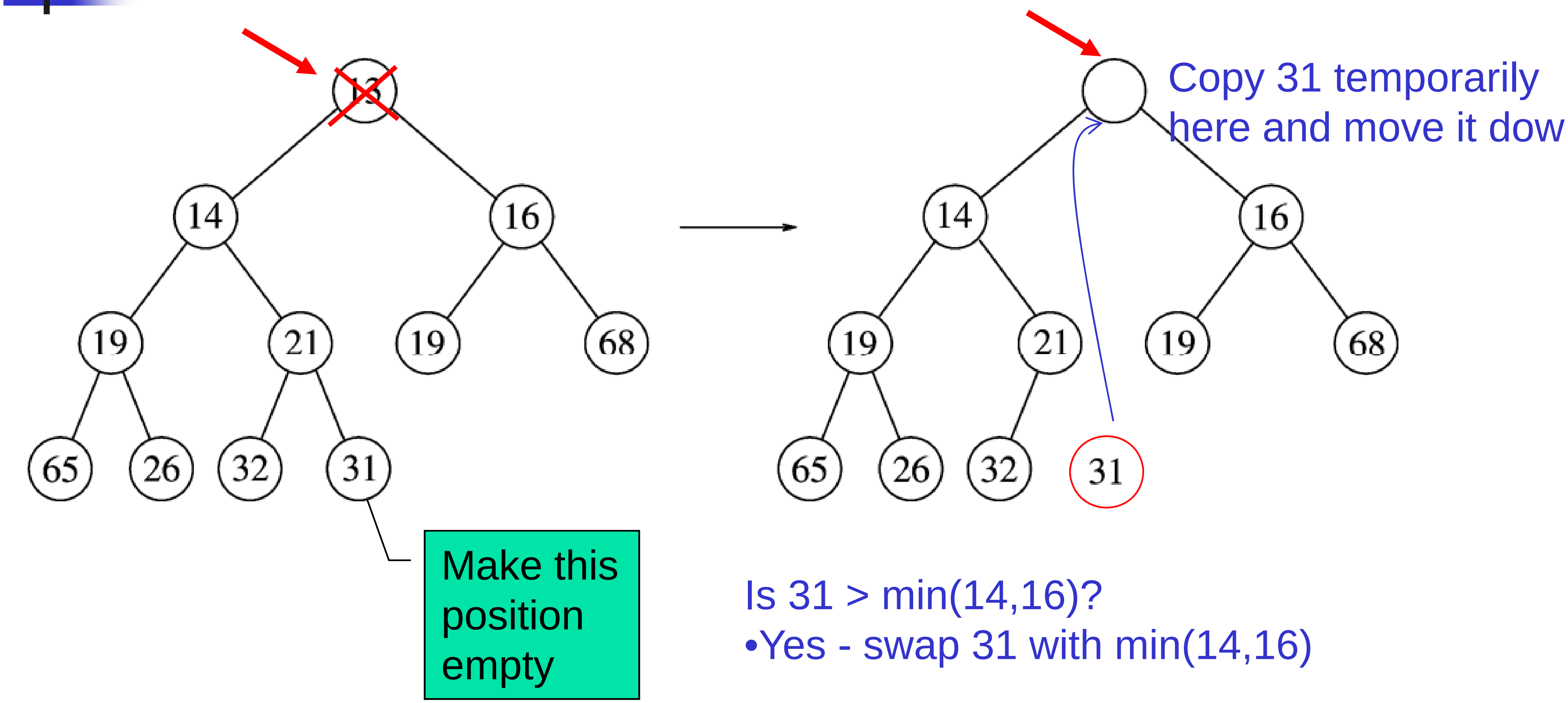
Percolating down…

Heap DeleteMin: Example



Percolating down…

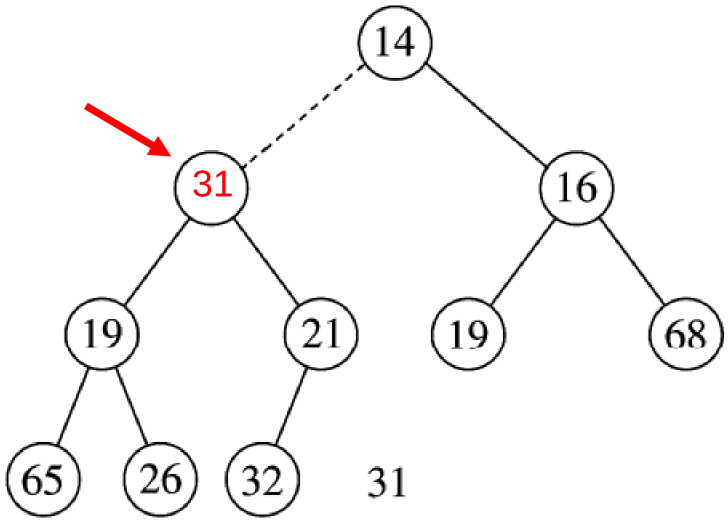
Heap DeleteMin: Example



Is 31 > min(14,16)?
- Yes - swap 31 with min(14,16)

Percolating down…

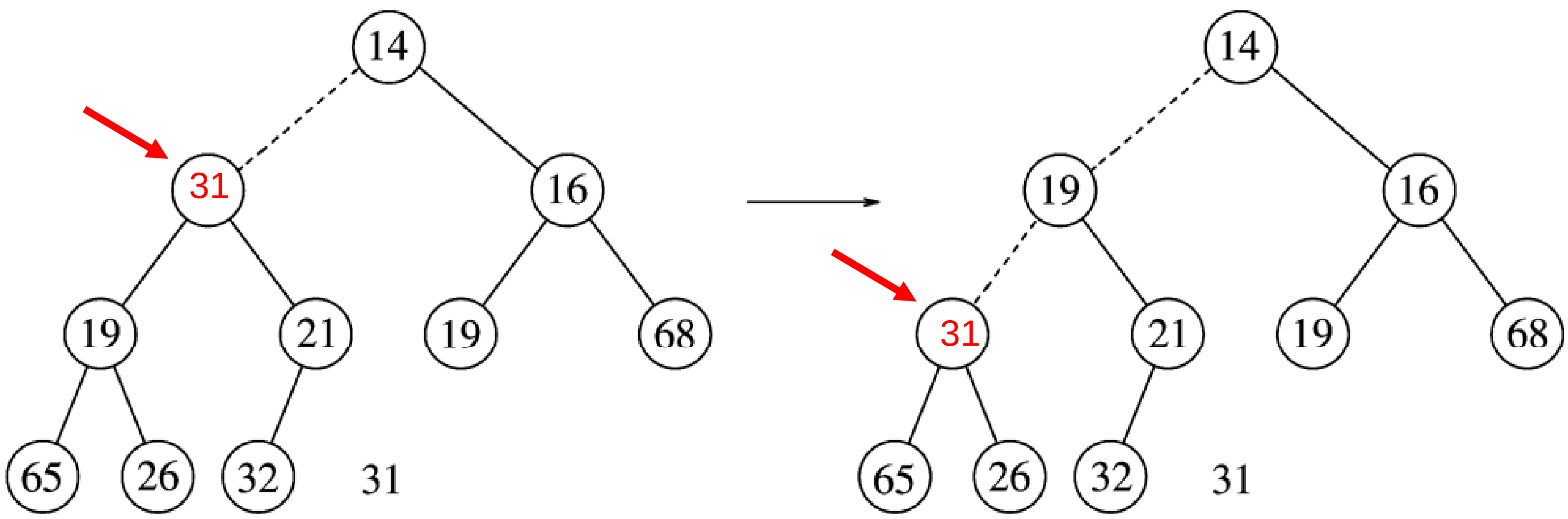
Heap DeleteMin: Example



Is 31 > min(19,21)?

•Yes - swap 31 with min(19,21)

Heap DeleteMin: Example



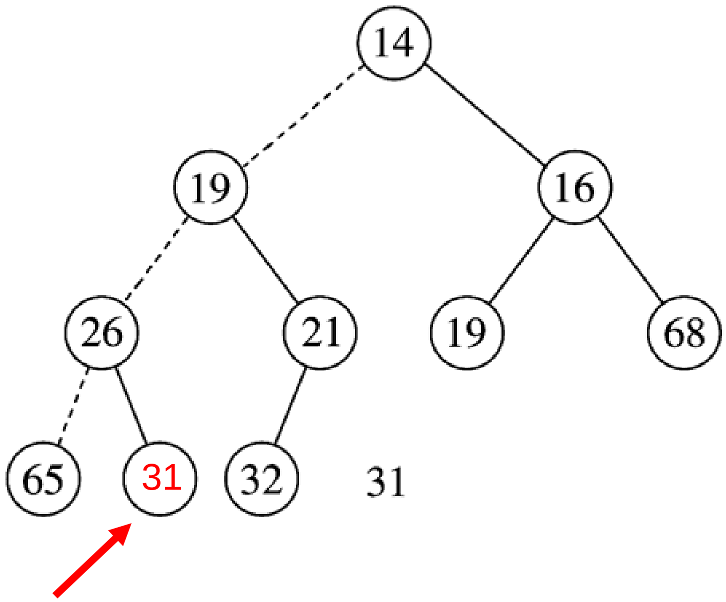
Is 31 > min(19,21)?
- Yes - swap 31 with min(19,21)

Is 31 > min(65,26)?
- Yes - swap 31 with min(65,26)

Percolating down…

Percolating down…

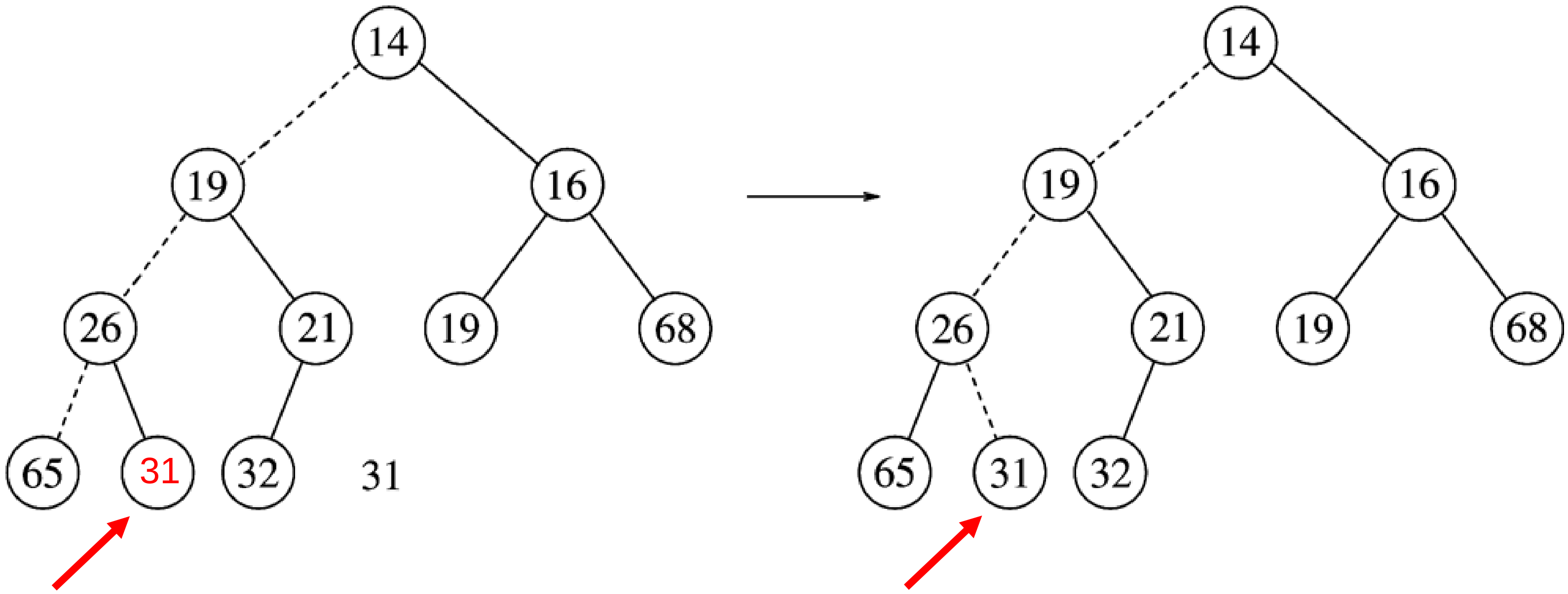
Heap DeleteMin: Example



Percolating down…

Percolating down…

Heap DeleteMin: Example



✓ Heap order prop

✓ Structure prop

Heap DeleteMin: Implementation

```
/**
 * Remove the minimum item.
 * Throws UnderflowException if empty.
 */
void deleteMin( )
{
    if( isEmpty( ) )
        throw UnderflowException( );

    array[ 1 ] = array[ currentSize-- ];
    percolateDown( 1 );
}
```

```
/**
 * Remove the minimum item and place it in minItem.
 * Throws UnderflowException if empty.
 */
void deleteMin( Comparable & minItem )
{
    if( isEmpty( ) )
        throw UnderflowException( );

    minItem = array[ 1 ];
    array[ 1 ] = array[ currentSize-- ];
    percolateDown( 1 );
}
```

O(log N) time

Heap DeleteMin: Implementation

```
/**
 * Internal method to percolate down in the heap.
 * hole is the index at which the percolate begins.
 */
void percolateDown( int hole )
{
    int child;
    Comparable tmp = array[ hole ];

    for( ; hole * 2 <= currentSize; hole = child )
    {
        child = hole * 2;
        if( child != currentSize && array[ child + 1 ] < array[ child ] )
            child++;
        if( array[ child ] < tmp )
            array[ hole ] = array[ child ];
        else
            break;
    }
    array[ hole ] = tmp;
}
```

Percolate down

Left child

Right child

Pick child to swap with

Other Heap Operations

- *decreaseKey(p,v)*
  - Lowers the current value of item p to new priority value v
  - Need to percolate up
  - E.g., promote a job
- *increaseKey(p,v)*
  - Increases the current value of item p to new priority value v
  - Need to percolate down
  - E.g., demote a job
- *remove(p)*
  - First, decreaseKey(p,-∞)
  - Then, deleteMin
  - E.g., abort/cancel a job

Run-times for all three functions?

O(lg n)

Improving Heap Insert Time

- What if all N elements are all available upfront?

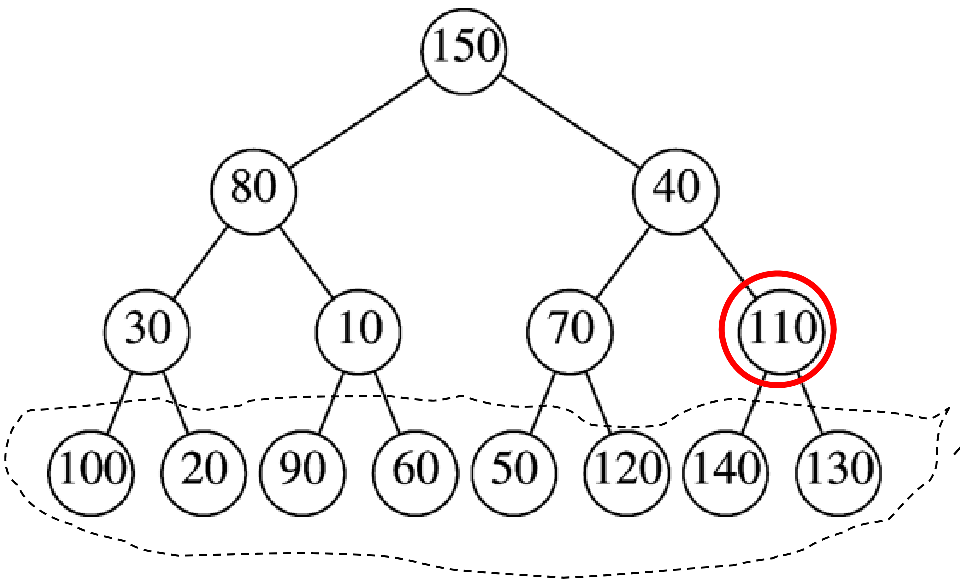
- To build a heap with N elements:
  - Default method takes O(N lg N) time
  - We will now see a new method called buildHeap() that will take O(N) time - i.e., optimal

Building a Heap

- Construct heap from initial set of N items
- Solution 1
  - Perform N inserts
  - $O(N \log_2 N)$ worst-case
- Solution 2 (use *buildHeap()*)
  - Randomly populate initial heap with structure property
  - Perform a percolate-down from each internal node (H[size/2] to H[1])
    - To take care of heap order property

BuildHeap Example

Input: { 150, 80, 40, 10, 70, 110, 30, 120, 140, 60, 50, 130, 100, 20, 90 }

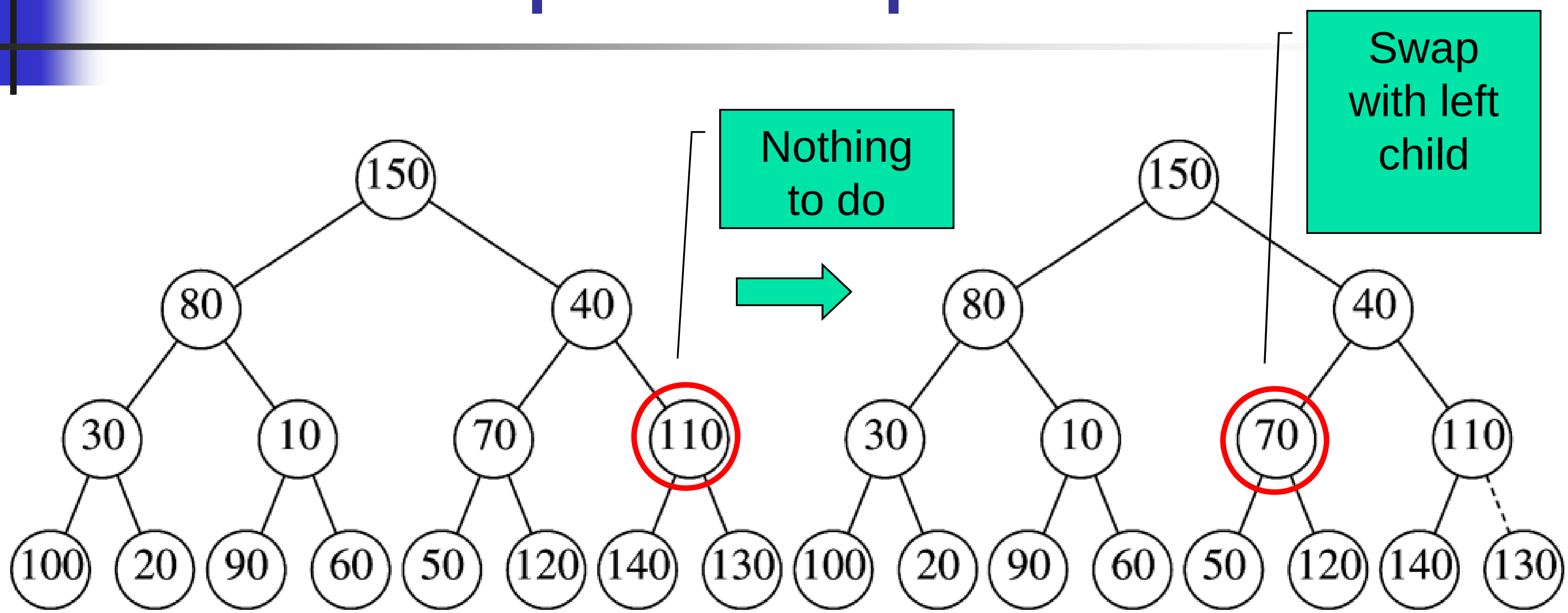


Leaves are all valid heaps (implicitly)

So, let us look at each internal node, from bottom to top, and fix if necessary

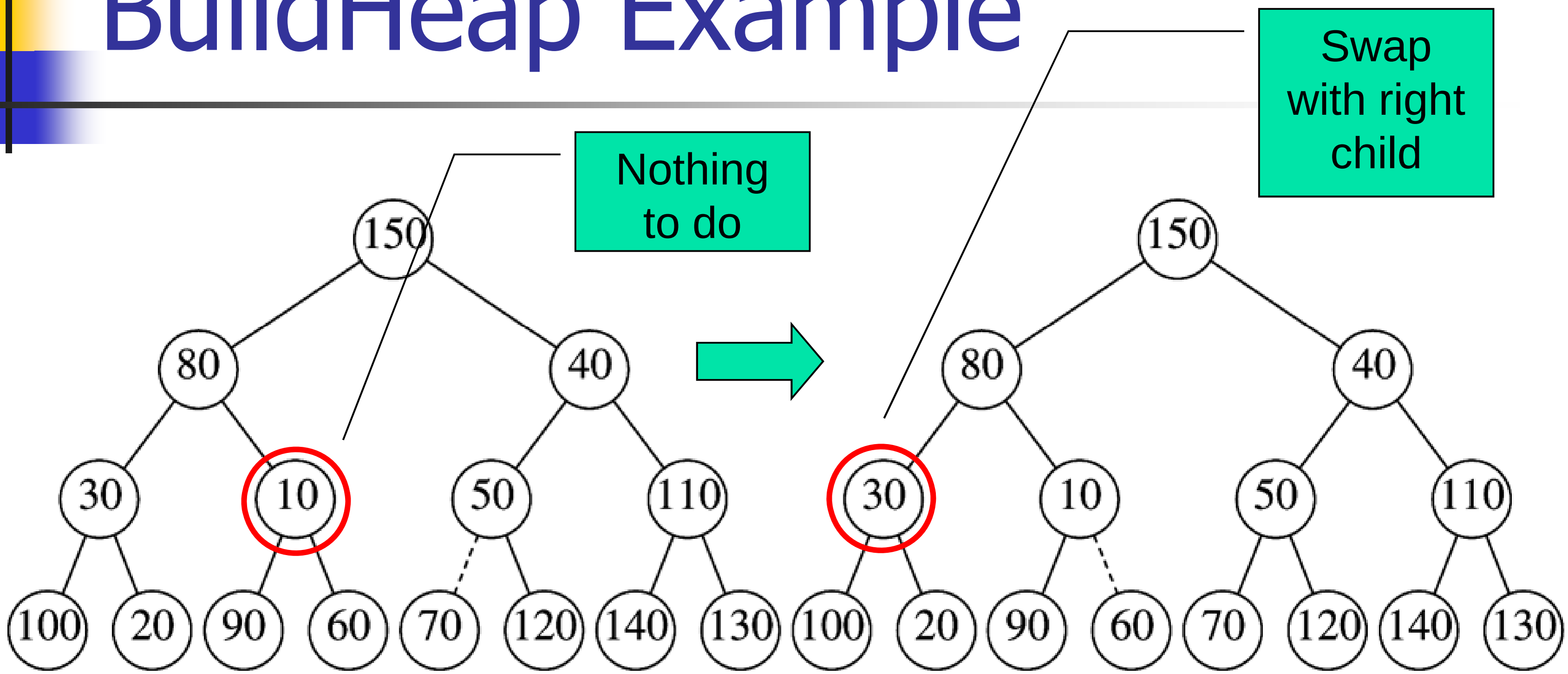
- Arbitrarily assign elements to heap nodes
- Structure property satisfied
- Heap order property violated
- Leaves are all valid heaps (implicit)

BuildHeap Example



- Randomly initialized heap
- Structure property satisfied
- Heap order property violated
- Leaves are all valid heaps (implicit)

BuildHeap Example

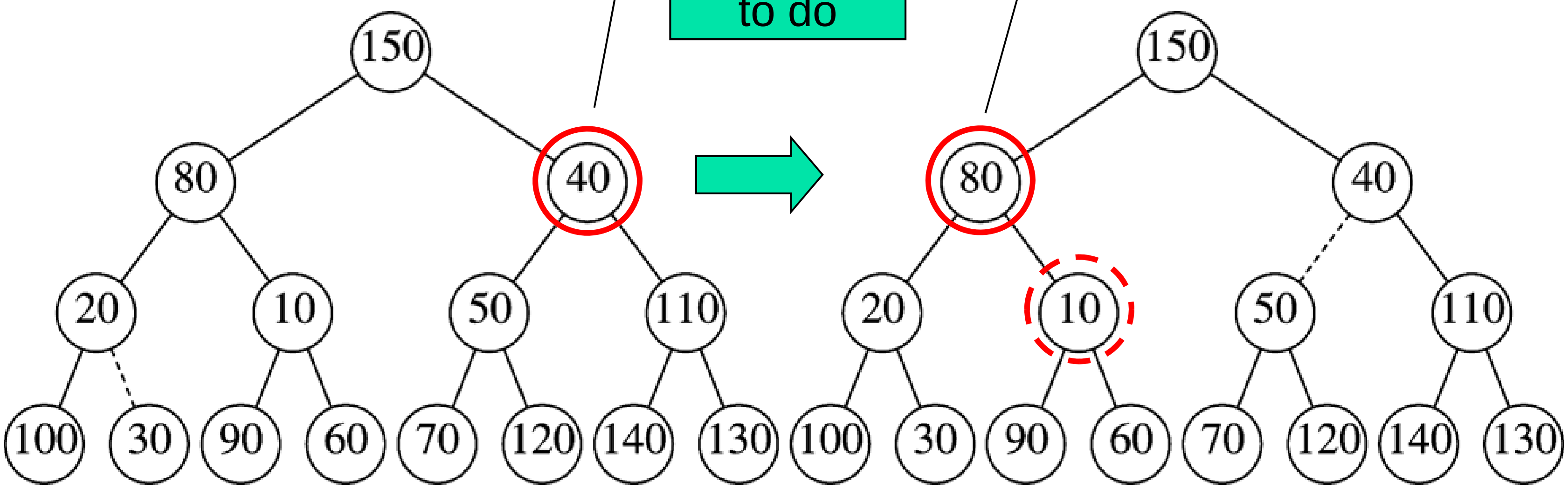


Dotted lines show path of percolating down

BuildHeap Example

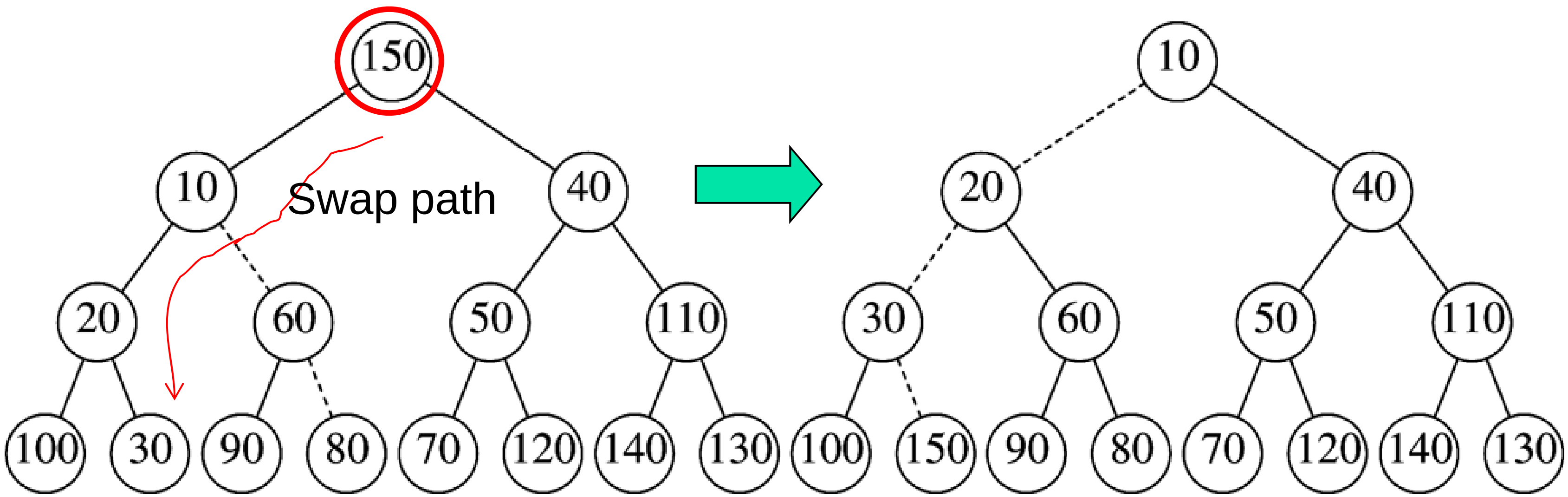
Nothing to do

Swap with right child & then with 60



Dotted lines show path of percolating down

BuildHeap Example



Final Heap

Dotted lines show path of percolating down

BuildHeap Implementation

```
explicit BinaryHeap( const vector<Comparable> & items )
  : array( items.size( ) + 10 ), currentSize( items.size( ) )
{
    for( int i = 0; i < items.size( ); i++ )
        array[ i + 1 ] = items[ i ];
    buildHeap( );
}

/**
 * Establish heap order property from an arbitrary
 * arrangement of items. Runs in linear time.
 */
void buildHeap( )
{
    for( int i = currentSize / 2; i > 0; i-- )
        percolateDown( i );
}
```

Start with lowest, rightmost internal node

BuildHeap() : Run-time Analysis

- Run-time = ?
  - O(sum of the heights of all the internal nodes) because we may have to percolate all the way down to fix every internal node in the worst-case
- <u>Theorem 6.1</u> **HOW?**
  - *For a perfect binary tree of height h, the sum of heights of all nodes is* $2^{h+1} - 1 - (h + 1)$
- Since $h = \lg N$, then sum of heights is O(N)
- Will be slightly better in practice

<u>Implication:</u> Each insertion costs O(1) amortized time

Build Heap analysis

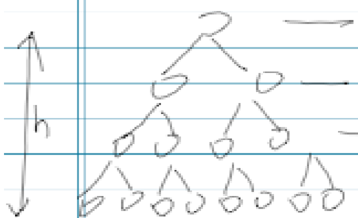
Thm: Runtime $= O(\text{sum of heights of all int nodes})$

$\equiv O(\text{sum of heights of all nodes})$ incl. leaves

$= O(n)$

Proof:

~~Runtime~~ Sum of heights of all nodes = sum of heights of nodes at each level



level	# nodes	height
0	2^0	$3 = (h-0)$
1	2^1	$2 = (h-1)$
2	2^2	$1 = (h-2)$
3	2^3	$0 = (h-3)$

$\therefore \sum$ heights of all nodes

$$= \sum_{i=0}^{h} (h-i) \times 2^i$$

$$= \sum_{i=0}^{h} h2^i - \sum_{i=0}^{h} i2^i$$

$$= h\sum_{i=0}^{h} 2^i - \sum_{i=0}^{h} i2^i$$

$$= h(2^{h+1}-1) - S$$

$$= h(2^{h+1}) - h - (h-1)2^{h+1} + 2$$

$$= 1\cdot 2^{h+1} - h + 2$$

$$= 2^{h+1} - h + 2$$

Let $S = \sum_{i=0}^{h} i2^i$

$\Rightarrow S = 1\cdot2^1 + 2\cdot2^2 + 3\cdot2^3 + 4\cdot2^4 + \dots + h2^h$

$2S = 1\cdot2^2 + 2\cdot2^3 + 3\cdot2^4 + 4\cdot2^5 + \dots + h2^{h+1}$

$2S - S = -[1\cdot2^1 + 2^2 + 2^3 + 2^4 + \dots + 2^h] + h2^{h+1}$

$\Rightarrow S = -[2^{h+1} - 1 - 1] + h2^{h+1}$

$= (h-1)2^{h+1} - 2$

As $h = \lg N$

$\Rightarrow \sum \text{heights} = 2^{\lg N + 1} - \lg N + 2$

$= 2N - \lg N + 2 = O(N)$

Binary Heap Operations Worst-case Analysis

- Height of heap is $\lfloor \log_2 N \rfloor$
- insert: O(lg N) for each insert
  - In practice, expect less
- buildHeap insert: O(N) for N inserts
- deleteMin: O(lg N)
- decreaseKey: O(lg N)
- increaseKey: O(lg N)
- remove: O(lg N)

Applications

- Operating system scheduling
  - Process jobs by priority
- Graph algorithms
  - Find shortest path
- Event simulation
  - Instead of checking for events at each time click, look up next event to happen

An Application: The Selection Problem

- Given a list of n elements, find the k^{th} smallest element

- Algorithm 1:
  - Sort the list => $O(n \log n)$
  - Pick the k^{th} element => $O(1)$
- A better algorithm:
  - Use a binary heap (minheap)

Selection using a MinHeap

- **Input: n elements**
- **Algorithm:**

1. buildHeap(n) ==> $O(n)$
2. Perform k deleteMin() operations ==> $O(k \log n)$
3. Report the k^{th} deleteMin output ==> $O(1)$

Total run-time = $O(n + k \log n)$

If $k = O(n/\log n)$ then the run-time becomes $O(n)$

Other Types of Heaps

- Binomial Heaps
- d-Heaps
  - Generalization of binary heaps (ie., 2-Heaps)
- Leftist Heaps
  - Supports merging of two heaps in o(m+n) time (ie., sub-linear)
- Skew Heaps
  - O(log n) amortized run-time
- Fibonacci Heaps

Run-time Per Operation

	Insert	DeleteMin	Merge (=H_1+H_2)
Binary heap	▪ O(log n) worst-case ▪ O(1) amortized for buildHeap	O(log n)	O(n)
Leftist Heap	O(log n)	O(log n)	O(log n)
Skew Heap	O(log n)	O(log n)	O(log n)
Binomial Heap	▪ O(log n) worst case ▪ O(1) amortized for sequence of n inserts	O(log n)	O(log n)
Fibonacci Heap	O(1)	O(log n)	O(1)

Priority Queues in STL

- Uses ***Binary heap***
- Default is MaxHeap
- Methods
  - Push, top, pop, empty, clear

```
#include <priority_queue>
int main ()
{
  priority_queue<int> Q;
  Q.push (10);
  cout << Q.top ();
  Q.pop ();
}
```

Calls DeleteMax()

For MinHeap: declare priority_queue as:

priority_queue<int, vector<int>, greater<int>> Q;

Refer to Book Chapter 6, Fig 6.57 for an example

Binomial Heaps

Binomial Heap

- A binomial heap is a *forest* of heap-ordered binomial trees, satisfying:

  i) Structure property, and

  ii) Heap order property

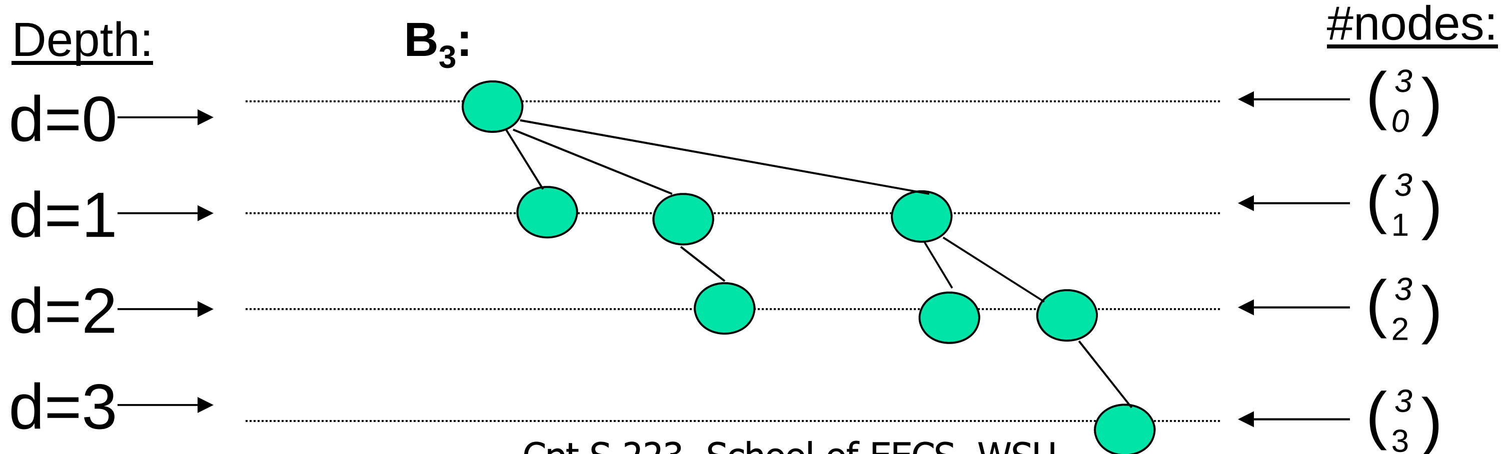
- A binomial heap is different from binary heap in that:
  - Its structure property is totally different
  - Its heap-order property (within each binomial tree) is the same as in a binary heap

Note: A binomial tree *need not* be a binary tree

Definition: A "Binomial Tree" B_k

- A binomial tree of height k is called B_k:
  - It has 2^k nodes
  - The number of nodes at depth $d = \binom{k}{d}$

$\binom{k}{d}$ is the form of the co-efficients in binomial theorem



What will a Binomial Heap with n=31 nodes look like?

- We know that:
  - i) A binomial heap should be a forest of binomial trees
  - ii) Each binomial tree has power of 2 elements
- So how many binomial trees do we need?

$$\begin{array}{c} \quad\quad\quad\quad\quad\quad\quad B_4\ B_3\ B_2\ B_1\ B_0 \\ \boxed{n = 31 = (1\ 1\ 1\ 1\ 1)_2} \end{array}$$

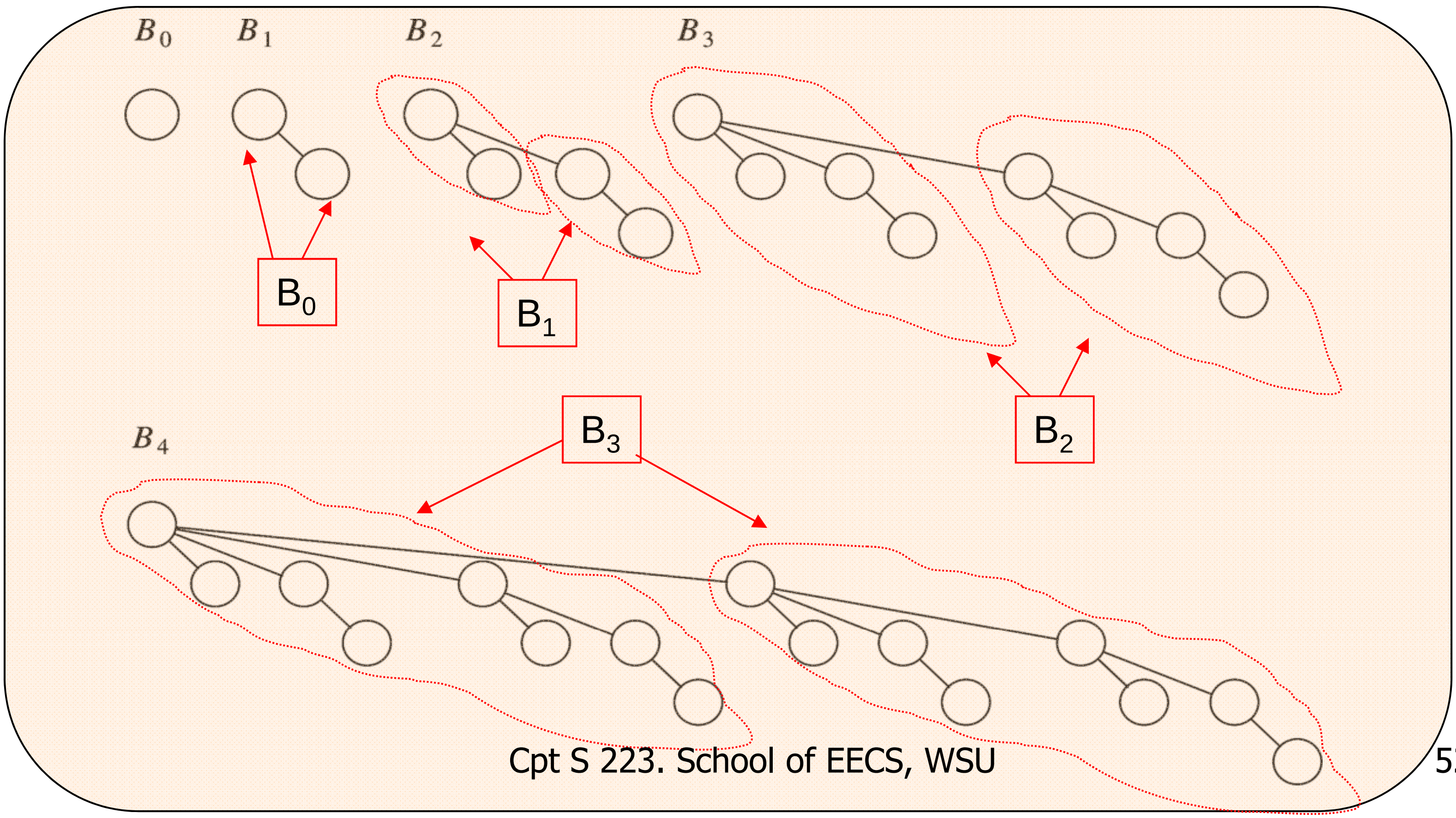
A Binomial Heap w/ n=31 nodes

$B_4\ B_3\ \ B_2\ B_1\ B_0$

$$n = 31 = (1\ 1\ 1\ 1\ 1)_2$$

$B_i == B_{i-1} + B_{i-1}$

Forest of binomial trees $\{B_0, B_1, B_2, B_3, B_4\}$



Binomial Heap Property

- Lemma: There exists a binomial heap for every positive value of *n*

- Proof:
  - All values of *n* can be represented in binary representation
    - Have one binomial tree for each power of two with co-efficient of 1
    - Eg., n=10 ==> $(1010)_2$ ==> forest contains $\{B_3, B_1\}$

Binomial Heaps: *Heap-Order Property*

- Each binomial tree should contain the minimum element at the root of every subtree
  - Just like binary heap, except that the tree here is a binomial tree structure (and not a complete binary tree)

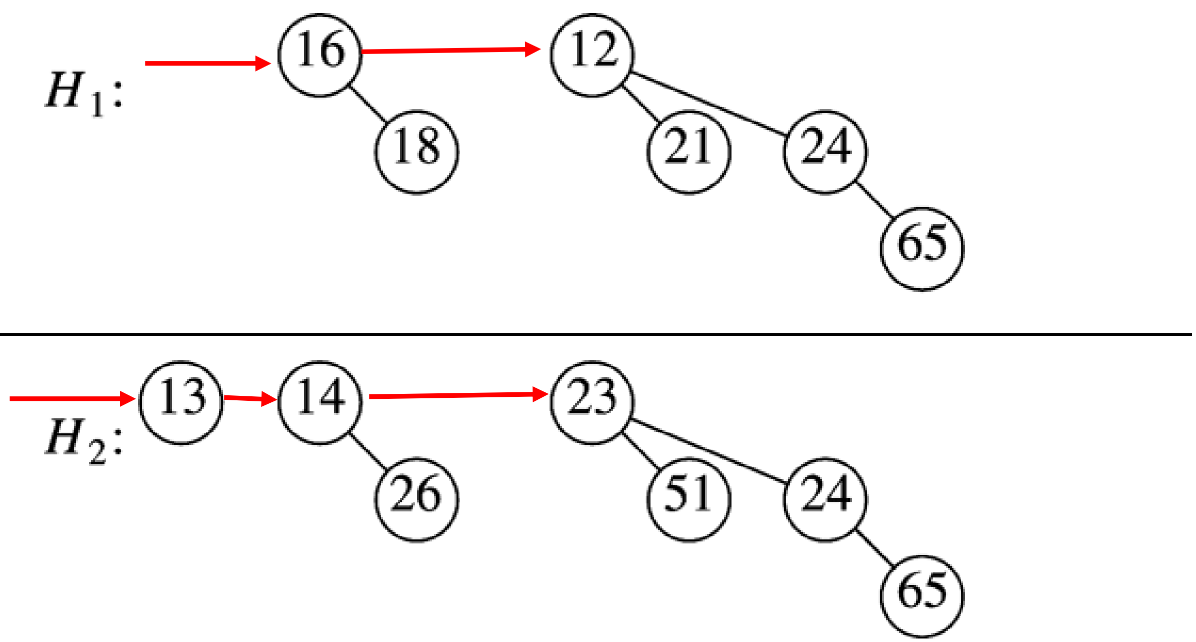
- The order of elements across binomial trees is irrelevant

Definition: Binomial Heaps

- A binomial heap of n nodes is:
  - (Structure Property) A forest of binomial trees as dictated by the binary representation of n
  - (Heap-Order Property) Each binomial tree is a min-heap or a max-heap

Binomial Heaps: Examples

Two different heaps:



Key Properties

- Could there be multiple trees of the same height in a binomial heap?

no

- What is the upper bound on the number of binomial trees in a binomial heap of *n* nodes?

$\lceil \lg n \rceil$

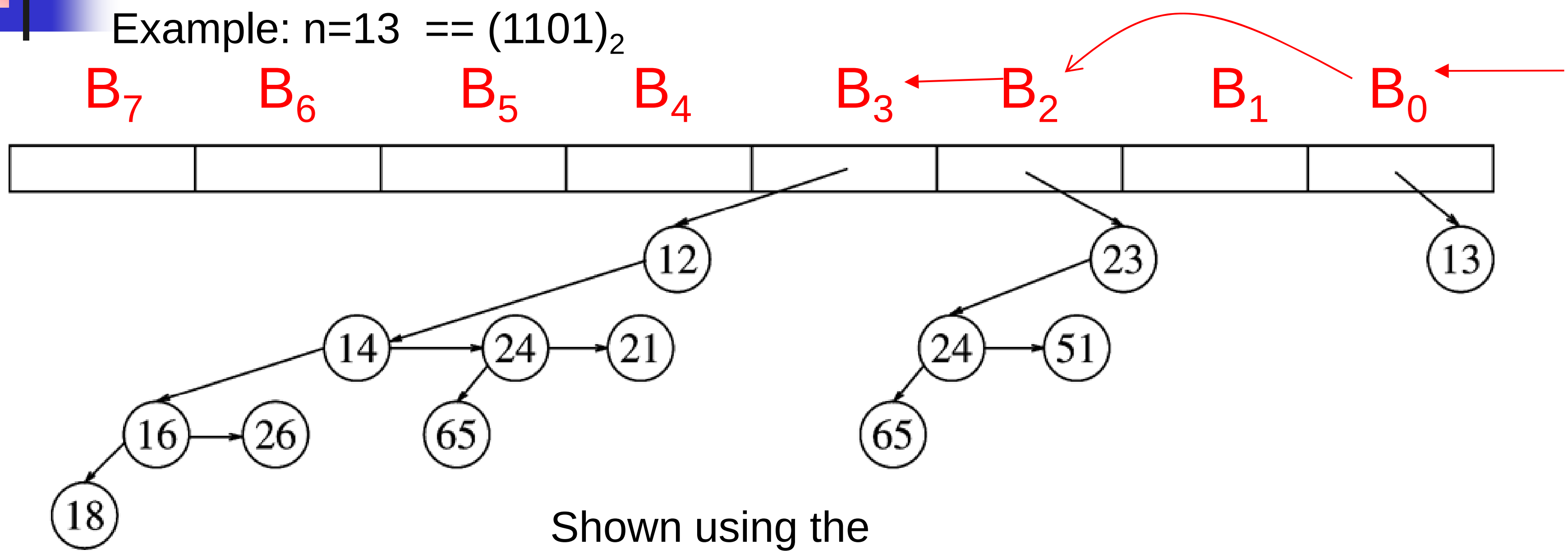
- Given *n*, can we tell (for sure) if B_k exists?

B_k exists if and only if:
the k^{th} least significant bit is 1
in the binary representation of n

An Implementation of a Binomial Heap

Maintain a linked list of
tree pointers (for the forest)

Example: n=13 == $(1101)_2$



Shown using the
left-child, right-sibling pointer method

Analogous to a bit-based representation of a binary number n

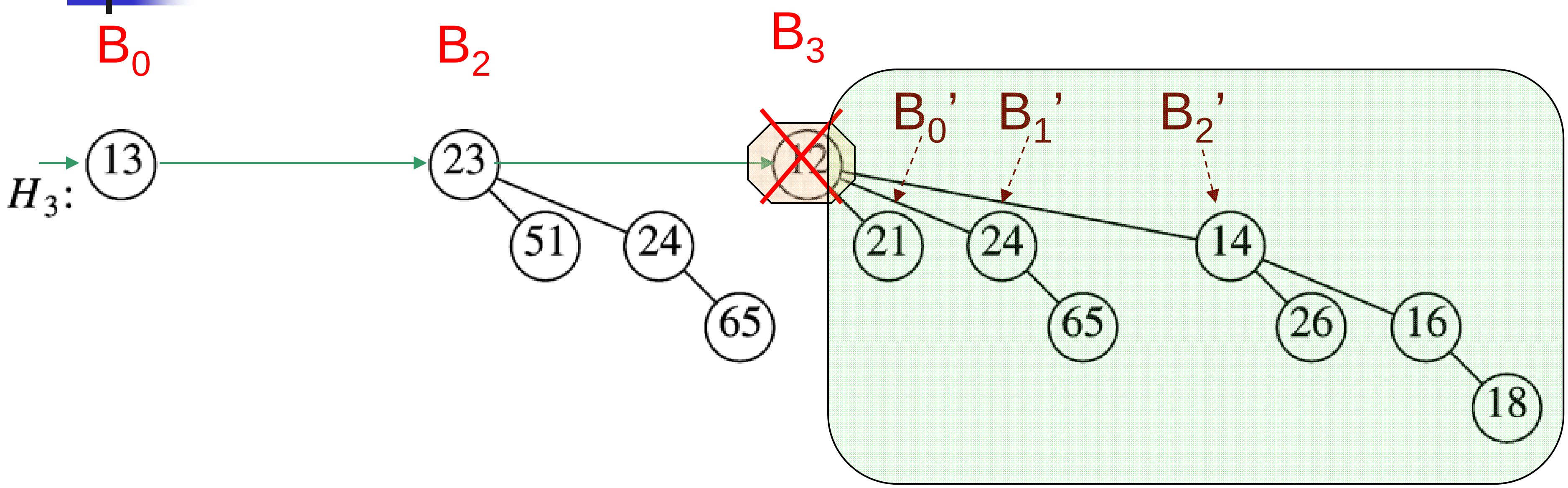
Binomial Heap: Operations

- x <= DeleteMin()
- Insert(x)
- Merge(H_1, H_2)

DeleteMin()

- **Goal:** Given a binomial heap, H, find the minimum and delete it
- **Observation:** The root of each binomial tree in H contains its minimum element
- **Approach:** Therefore, return the minimum of all the roots (minimums)
- **Complexity:** O(log n) comparisons
  (because there are only O(log n) trees)

FindMin() & DeleteMin() Example



For DeleteMin(): After delete, how to adjust the heap?

New Heap : Merge { B_0, B_2 } & { B_0', B_1', B_2' }

Insert(x) in Binomial Heap

- **Goal:** To insert a new element x into a binomial heap H
- **Observation:**
  - Element x can be viewed as a single element binomial heap
    - => Insert (H,x) == **Merge(H, {x})**

So, if we decide how to do merge we will automatically figure out how to implement both insert() and deleteMin()

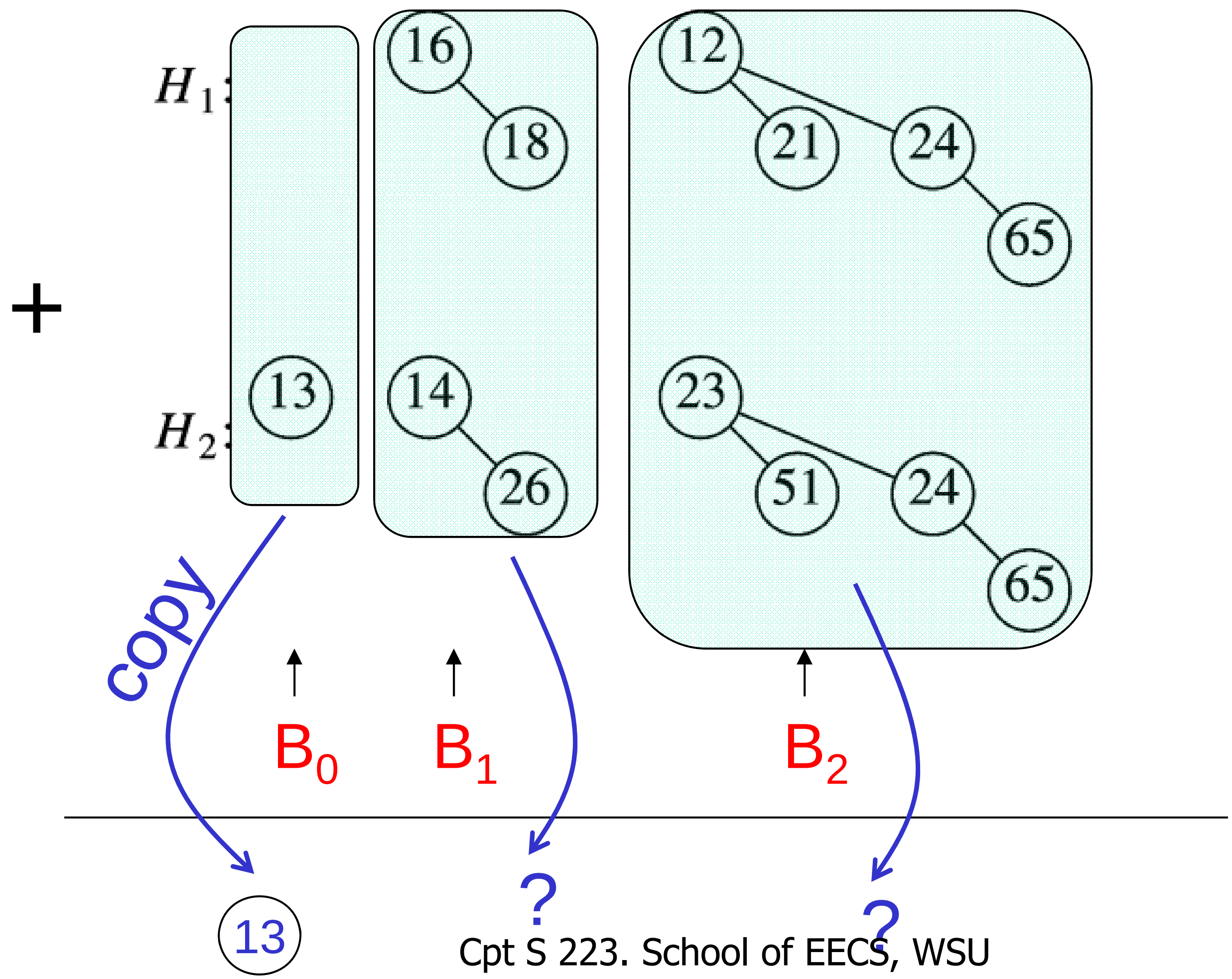
Merge(H_1,H_2)

- Let n_1 be the number of nodes in H_1
- Let n_2 be the number of nodes in H_2
- Therefore, the new heap is going to have $n_1 + n_2$ nodes
  - Assume $n = n_1 + n_2$

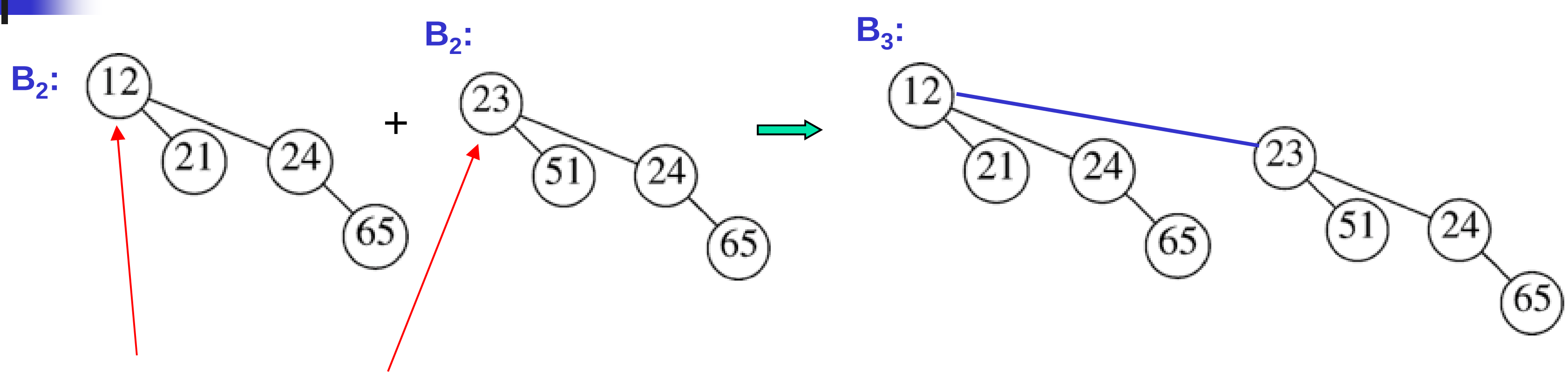
- Logic:
  - Merge trees of same height, starting from lowest height trees
  - If only one tree of a given height, then just copy that
  - Otherwise, need to do carryover (just like adding two binary numbers)

Idea: merge tree of same heights

Merge: Example



How to Merge Two Binomial Trees of the *Same* Height?

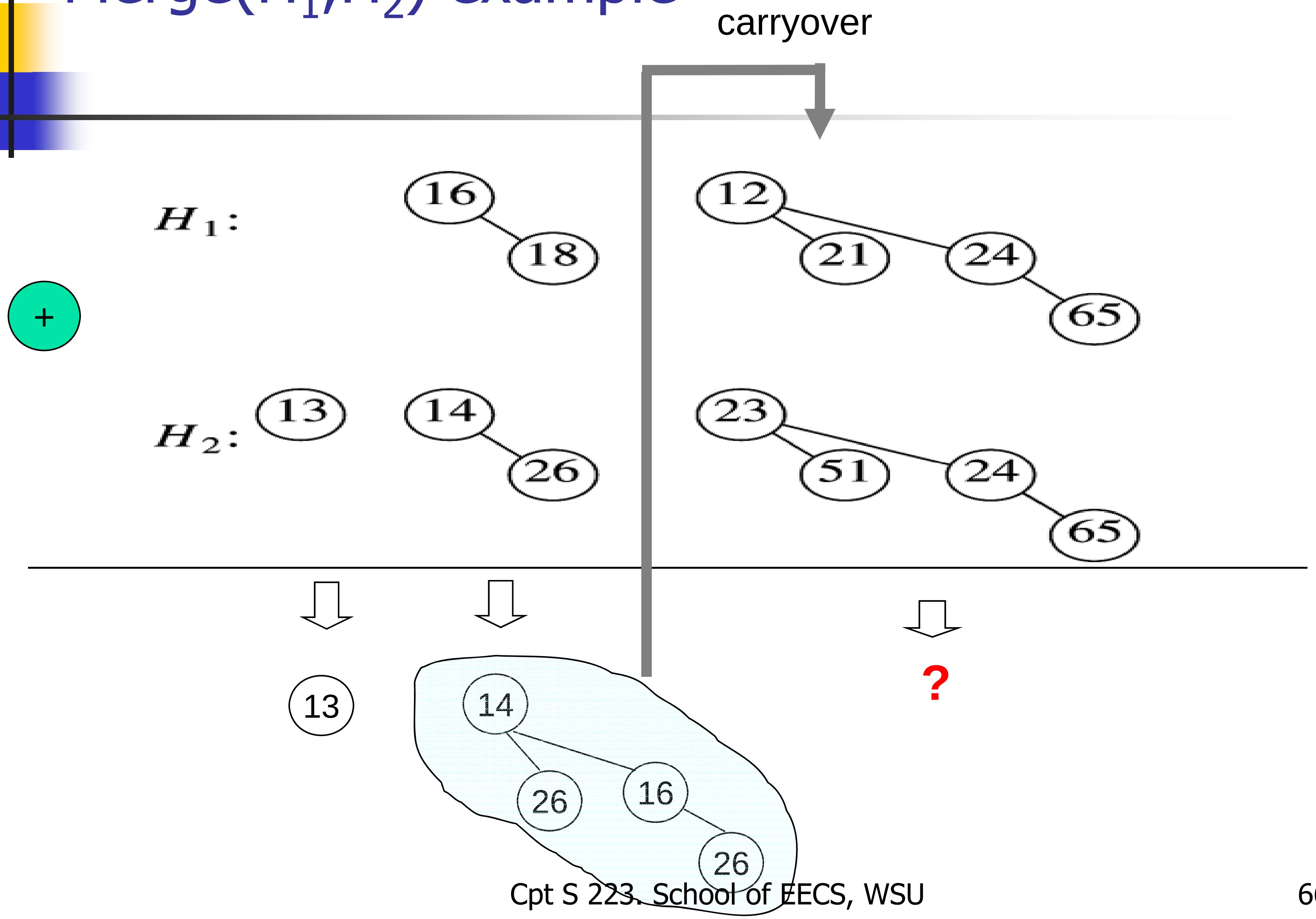


Simply compare the roots

```
/**
 * Return the result of merging equal-sized t1 and t2.
 */
BinomialNode * combineTrees( BinomialNode *t1, BinomialNode *t2 )
{
    if( t2->element < t1->element )
        return combineTrees( t2, t1 );
    t2->nextSibling = t1->leftChild;
    t1->leftChild = t2;
    return t1;
}
```

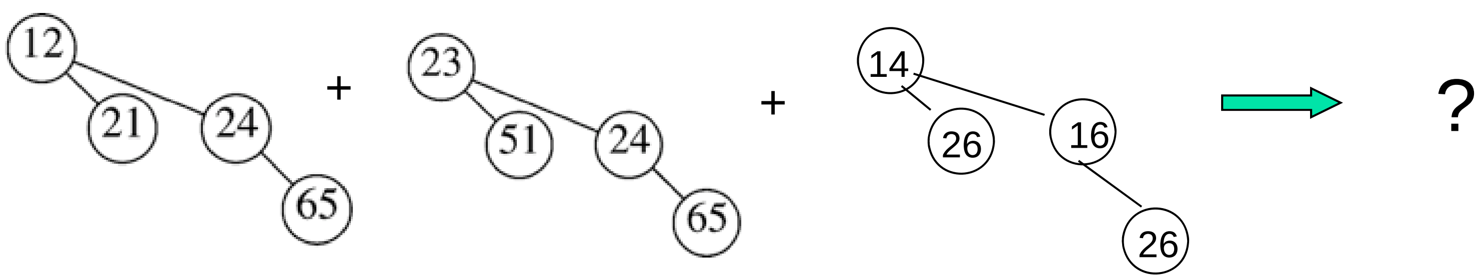
Note: Merge is defined for only binomial trees with the <u>same height</u>

Merge(H_1,H_2) example



How to Merge *more than two* binomial trees of the same height?

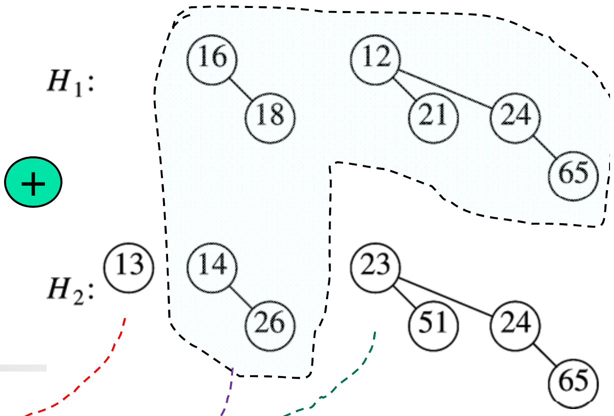
- Merging more than 2 binomial trees of the same height could generate carry-overs



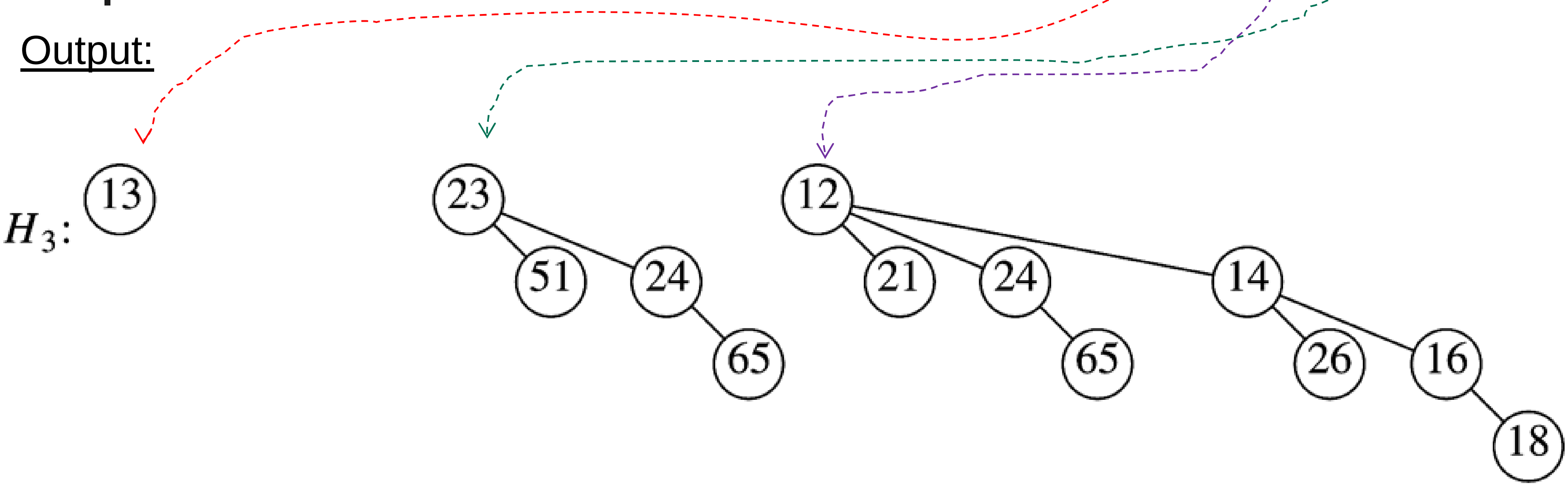
Merge any two and leave the third as carry-over

Merge(H_1,H_2) : Example

Input:



Output:



There are two other possible answers

Merge cost $\propto$ $\log(\max\{n_1,n_2\}) = O(\log n)$ comparisons

Run-time Complexities

- Merge takes O(log n) comparisons

- <u>Corollary:</u>
  - Insert and DeleteMin also take O(log n)

- *It can be further proved that an uninterrupted sequence of **m** Insert operations takes only **O(m)** time per operation, implying **O(1) amortize time per insert***
  - <u>Proof Hint:</u>
    - For each insertion, if *i* is the least significant bit position with a 0, then number of comparisons required to do the next insert is *i+1*
    - If you count the #bit flips for each insert, going from insert of the first element to the insert of the last (n^{th}) element, then
      => amortized run-time of O(1) per insert

unaffected

affected

```
10010111
       1
-------------
10011000
```

Binomial Queue Run-time Summary

- Insert
  - O(lg n) worst-case
  - O(1) amortized time if insertion is done in an uninterrupted sequence (i.e., without being intervened by deleteMins)
- DeleteMin, FindMin
  - O(lg n) worst-case
- Merge
  - O(lg n) worst-case

Run-time Per Operation

	Insert	DeleteMin	Merge (=H_1+H_2)
Binary heap	▪ O(log n) worst-case ▪ O(1) amortized for buildHeap	O(log n)	O(n)
Leftist Heap	O(log n)	O(log n)	O(log n)
Skew Heap	O(log n)	O(log n)	O(log n)
Binomial Heap	▪ O(log n) worst case ▪ O(1) amortized for sequence of n inserts	O(log n)	O(log n)
Fibonacci Heap	O(1)	O(log n)	O(1)

Summary

- Priority queues maintain the minimum or maximum element of a set
- Support O(log N) operations worst-case
  - insert, deleteMin, merge
- Many applications in support of other algorithms