Reduced Mass - An Alternative View

Jayeeta Chowdhury* and Arup Roy †

Department of Physics, Scottish Church College, Kolkata-6, India.

Date of Submission: 24th November, 2016 Date of Acceptance: 30th November, 2016

Abstract

Concept of reduced mass is very useful in the study of two-particle systems under mutual interaction. When one particle is being observed from the position of the other, it is the reduced mass of the system, which is in motion under mutual forces. We have pointed out that the special form of the equation of motion under such a condition is nothing but the effect of pseudo force. Using the concept of pseudo force we have extended this idea to many-particle systems. We have constructed a general equation using which one can easily find the acceleration of any particle observed by any other within an isolated many-particle system.

Key words: Two or many body problem, Reduced mass, Non-inertial frame, Pseudo force

1. Introduction

Two-body problems (i.e. two particles under the mutual interaction) are very common in Physics. In classical Physics, in particular, a system comprising of two particles under mutual interaction is of specific interest. There are many such systems, e.g., two mass points connected by a spring, binary stars, two charges separated by a distance and so on. In absence of external forces, the centre of mass of these systems moves with constant velocity. Introduction of the reduced mass of the system helps to analyse the motion using a single equation. The concept of reduced mass becomes more interesting when we note the role played by the pseudo forces. This new approach can be easily extended to systems having more than two particles under mutual interaction.

2. The Usual Approach to Study Two-body Problems

The ususal way in which we solve the two-body problem is the following:

O is the position of some inertial observer. With respect to O the position of the ith particle (with mass m_i) is $\vec{r_i}$, i can have values 1 and 2. In absence of external force, in a binary system, one particle is being attracted by the other along the line joining them. Let us denote the force on the ith particle due to the jth particle by $\vec{F_{ij}}$ $(i, j = 1, 2)$. Obviously $\vec{F_{ii}} = 0$ and $\vec{F_{ij}} = -\vec{F_{ji}}$. Equations of motions for the particles 1 and 2 are

*email: jyt.cdr@gmail.com, jcphys@scottishchurch.ac.in

†Retired, email: aryscottish@gmail.com

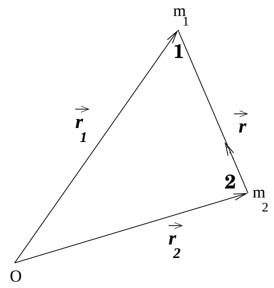

Figure 1: Schematic diagram of a two-particle system

$$m_1 \frac{d^2\vec{r_1}}{dt^2} = \vec{F_{12}} = F\hat{r}, \tag{1}$$

$$\text{and} \quad m_2 \frac{d^2\vec{r_2}}{dt^2} = \vec{F_{21}} = -F\hat{r} \qquad \text{say} \tag{2}$$

respectively. $\hat{r}$ is the unit vector from 2 to 1. $\vec{r} = r\hat{r} = \vec{r_1} - \vec{r_2}$. F is a function of m_1, m_2 and r in general; $\vec{F_{12}}$ is directed along $\hat{r}$. From eqns. (1) and (2) we can write

$$\frac{d^2\vec{r_1}}{dt^2} - \frac{d^2\vec{r_2}}{dt^2} = F\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\hat{r}$$

$$\text{or} \quad \ddot{\vec{r}} = \frac{F}{\mu}\hat{r} \qquad \text{say} \tag{3}$$

where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is called the reduced mass for the system[1;2].

Now $\vec{r}$ is the position vector of the 1st particle with respect to the 2nd particle. Eqn. (3) is the equation of motion of the 1st particle as seen by the 2nd particle: as if reduced mass μ is situated at the position of the particle 1 and its motion is being observed from the position of the particle 2. Motion of the reduced mass is under the mutual interaction between the two. While handling the two body problem we deal with this single equation and mathematics become simple.

However in eqn. (3) observer is the 2nd particle, and on it there is a force due to the 1st. So the observer is non-inertial. Let us approach the same problem from a different view point, by introducing the concept of pseudo forces.

3. An Accelerating Frame of Reference

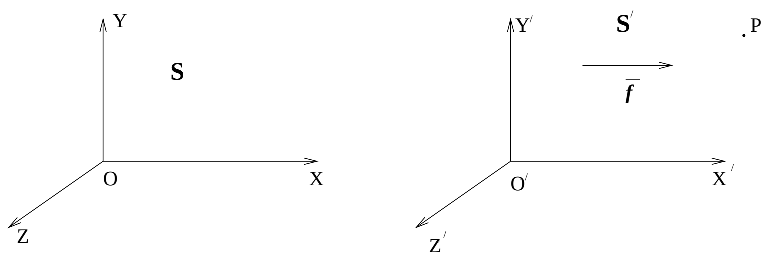

Figure 2: Two reference frames. S: inertial, S′: non-inertial.

Let us consider two reference frames S and S′ with origins at O and O' respectively. They have axes parallel to each other. At time $t = 0$, O and O' coincide. The frame S is inertial. The other one S′

is moving towards right with uniform acceleration $\vec{f}$ along X axis with respect to S (The choice is a general one, because space is isotropic.), hence S$'$ frame is non-inrtial.
Position of point P as described by S is

$$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z, \tag{4}$$

and the same as described by S$'$ is

$$\vec{r'} = \hat{i}x' + \hat{j}y' + \hat{k}z', \tag{5}$$

where

$$\begin{aligned} x' &= x - \frac{1}{2}ft^2, \\ y' &= y, \\ z' &= z. \end{aligned} \tag{6}$$

So velocity of P as measured by S$'$

$$\begin{aligned} \vec{v'} &= \hat{i}(\dot{x} - ft) + \hat{j}\dot{y} + \hat{k}\dot{z} \\ &= \vec{v} - \hat{i}ft, \end{aligned} \tag{7}$$

where $\vec{v}$ is the velocity of P as measured by S. Acceleration of P in S and S$'$ frame, $\vec{a}$ and $\vec{a'}$ respectively, are connected in the following way:

$$\begin{aligned} \vec{a'} &= \hat{i}(\ddot{x} - f) + \hat{j}\ddot{y} + \hat{k}\ddot{z} \\ &= \vec{a} - \hat{i}f \\ &= \vec{a} - \vec{f}. \end{aligned} \tag{8}$$

Now in a non-inertial frame if we define force as mass times acceleration, then force on a particle of mass m situated at P is

$$\vec{F_{NI}} = m(\vec{a} - \vec{f}) = m\vec{a} - m\vec{f}, \tag{9}$$

where $m\vec{a}$ is the inertial force and $-m\vec{f}$ is the pseudo force that comes into play due to the acceleration of the reference frame.

4. The New Approach to Study Two-body Problem

Let us again consider the two-body problem. Inertial observer is at O (Fig. 1). With respect to it acceleration of particle 2 is

$$\frac{d^2\vec{r_2}}{dt^2} = \vec{a_2} \qquad \text{say} \tag{10}$$

Therefore for the non-inertial observer at position 2 force on particle 1 is (refer eqn. (9))

$$\vec{F_{NI}} = \vec{F_{12}} - m_1\vec{a_2}, \tag{11}$$

where $\vec{F_{12}}$ is the force on the 1st particle due to the 2nd particle and $-m_1\vec{a_2}$ is the pseudo force arising due to the acceleration of 2nd particle. Now

$$\vec{a_2} = \frac{d^2\vec{r_2}}{dt^2} = \frac{\vec{F_{21}}}{m_2} = -\frac{\vec{F_{12}}}{m_2}. \tag{12}$$

Therefore, using eqn. (12) in eqn. (11), force on particle 1 as measured by particle 2 is

$$\vec{F_{12}} - m_1\left(-\frac{\vec{F_{12}}}{m_2}\right) = \vec{F_{12}}\left(1+\frac{m_1}{m_2}\right).$$

So, equation of motion of particle 1 as described by particle 2 is

$$\begin{aligned} m_1\frac{d^2\vec{r}}{dt^2} &= \vec{F_{12}}\left(1+\frac{m_1}{m_2}\right) \\ \text{or} \qquad \mu\ddot{\vec{r}} &= \vec{F_{12}} \end{aligned} \tag{13}$$

We observe that eqn. (14) is identical with eqn. (3). From the above derivation it is clear that the special form of the eqn. (14) is actually the effect of pseudo force.

5. Extension to N-particle System

We can extend this idea to cases where more than two particles under mutual interactions are involved. Fig. (3) shows such a three-particle system.

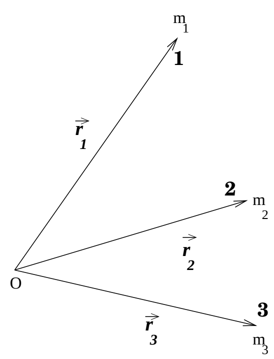

Figure 3: Schematic diagram of a three-particle system

According to inertial observer O, position of mass m_i is given by $\vec{r_i}$ $(i = 1, 2, 3)$. Acceleration of 1 is

$$\vec{a_1} = \frac{1}{m_1}(\vec{F_{12}} + \vec{F_{13}}) \tag{14}$$

where $\vec{F_{ji}}$ is the force on j due to i. When 1 observes the motion of 2, force on 2 is the sum of inertial forces and pseudo forces:

$$
\begin{aligned}
&\underbrace{(\vec{F_{21}} + \vec{F_{23}})}_{inertial\ force} \underbrace{-m_2 \frac{1}{m_1}(\vec{F_{12}} + \vec{F_{13}})}_{pseudo\ force} \\
&= \vec{F_{21}} \left(1 + \frac{m_2}{m_1}\right) + \vec{F_{23}} - \frac{m_2}{m_1}\vec{F_{13}} \qquad (15)
\end{aligned}
$$

So,

$$
\underbrace{\ddot{\vec{r_{21}}} = \frac{\vec{F_{21}}}{\mu}}_{as\ in\ 2\ body\ system\ (1-2)} + \underbrace{\frac{1}{m_2}\vec{F_{23}}}_{acc.\ of\ 2\ due\ to\ 3} - \underbrace{\frac{1}{m_1}\vec{F_{13}}}_{acc.\ of\ 1\ due\ to\ 3} \qquad (16)
$$

where $\ddot{\vec{r_{ij}}}$ is the acceleration of i measured by j. Similarly, for a four-particle system, equation of motion of 2 as described by 1 will be

$$
\underbrace{\ddot{\vec{r_{21}}} = \frac{\vec{F_{21}}}{\mu}}_{as\ in\ 2\ body\ system\ (1-2)} + \underbrace{\frac{1}{m_2}(\vec{F_{23}} + \vec{F_{24}})}_{acc.\ of\ 2\ due\ to\ 3,4} - \underbrace{\frac{1}{m_1}(\vec{F_{13}} + \vec{F_{14}})}_{acc.\ of\ 1\ due\ to\ 3,4} \qquad (17)
$$

Thus for an N-particle system under mutual interaction one can find the acceleration of any particle (i) as seen by any other (j) and can express this statement in the following form:

$$
\ddot{\vec{r_{ij}}} = \frac{F_{ij}}{\mu_{ij}} + \frac{1}{m_i}\sum_{k\neq i,j} \vec{F_{ik}} - \frac{1}{m_j}\sum_{k\neq i,j} \vec{F_{jk}}, \qquad (18)
$$

where $\mu_{ij} = \frac{m_i m_j}{m_i + m_j}$. If all the masses and detailed geometry of the system is known, $\ddot{\vec{r_{ij}}}$ can be exactly ascertained.

6. Conclusion

Mapping of a two-body problem into two one body (centre of mass and reduced mass) problems was very common and useful technique in classical Physics. Surprisingly, reduced mass contains in itself the mass of the particle being observed, as well as the mass of the observer. Our very aim was to find the basic Physics behind this special form of reduced mass. The develpoment of eqn. (14) from eqn. (9) using the concept of pseudo force clarifies why the form is so. It is the non-inertial observer who is playing the crucial role. We have further extended our study to isolated systems with many particles under mutual interaction and formed a general equation (eqn. (19)) using which one can predict the motion of one particle while being seated on any other.

References

[1] Goldstein H 2001 Classical Mechanics (Narosa Publishing House, New Delhi)

[2] Kittel C, Knight W D, Ruderman M A, Helmholz C A and Moyer B J 2008 Mechanics (Tata McGraw-Hill Publishing Company Limited, New Delhi)